EPYMT TDG Group 2 Tutorial 1 Solution

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1 Standard notation

We will usually use the following notations in the future without any explanation (That's common in university Mathematics):

i $x_1, x_2, ..., x_i, ..., x_n$:
x_i means the $i-$th variable/unknown. In university Mathematics, we need to deal with problems handling lots of unknowns. It is impossible to let something like x, y, z as we don't know the number of variables we are going to let. So we have to use x_i instead for better index.

ii $\mathbf{v}$ or $\overrightarrow{v}$: A vector .
Don't try to bold the alphabet and say it as a vector in your test or exams. It is very hard for us to read it.

iii Notation of set:
$A = \{x : x \text{ satisfies property P}\}$ is a set A containing all elements x such that (That's the meaning of ":" in the above) x satisfies property P.
E.g.

(1) $\{x : x \text{ is even}\} = \{..., -4, -2, 0, 2, 4, ...\}$

iv $x \in A$: x belongs to a set A

v $\forall$: For any ...

vi $\exists$: There exists **at least one/some** ...

vii $\exists!$: There exists **unique** ....

viii $\mathbb{R}$: Set of all real numbers.

ix $\mathbb{R}^n$: Set of all vectors in $n-$dimensions: $\{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in \mathbb{R}\}$

x $\mathbb{D}^1$: Set of all points on a unit disc: $\{(x, y) : x^2 + y^2 \leq 1\}$.

xi $\mathbb{S}^1$: Set of all points on the circumference of a unit circle: $\{(x, y) : x^2 + y^2 = 1\}$.

xii $\mathbb{S}^2$: Set of all points on the surface of a unit sphere: $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.

You can freely use these notations in homework/test/exam. But you have to use it correctly. Otherwise, marks may be deducted.

Please be reminded, you need to state clearly notations/symbols designed by you in the beginning of each problem. E.g.:

i Let S be the surface described in the question.

ii Let $A = \{\mathbf{a}, \mathbf{b}\}$ be a basis of the set S described in the question.

2 Integration techniques

For several standard formulae for integration, please refer to the lecture notes.

i Integration by substitution

(1) Trigo substitution:

Example: Evaluate

(a) $\displaystyle\int \frac{1}{x^2+1}\,dx.$

(b) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx.$

(c) $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx.$

ii Partial fraction decomposition

Example: Compute $\displaystyle\int \frac{1}{x^2-1}\,dx.$

iii Completing square

Example: Compute $\displaystyle\int \frac{1}{x^2+2x+2}\,dx.$

iv t-substitution

Let $t = \tan\frac{x}{2}$. Then we have

(1) $\sin x =$

(2) $\cos x =$

(3) $\tan x =$

(1)

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2}\\ &= 2\cdot\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2}\\ &= 2t\cdot\frac{1}{\sec^2\frac{x}{2}}\\ &= \frac{2t}{1+\tan^2\frac{x}{2}}\\ &= \frac{2t}{1+t^2}\end{aligned}$$

(2)

$$\begin{aligned}\cos x &= 2\cos^2\frac{x}{2} - 1\\ &= \frac{2}{\sec^2\frac{x}{2}} - 1\\ &= \frac{2}{1+\tan^2\frac{x}{2}} - 1\\ &= \frac{2}{1+t^2} - 1\\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$

(3) By formula, we have

$$\begin{aligned}\tan x &= \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}\\ &= \frac{2t}{1-t^2}\end{aligned}$$

Or we can simply use the above two results:

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x}\\ &= \frac{2t}{1+t^2} \div \frac{1-t^2}{1+t^2}\\ &= \frac{2t}{1-t^2}\end{aligned}$$

Example: Evaluate $\int \frac{1}{2+\sin x}\,dx$:

(You may also try to use this method to compute $\int \frac{1}{1+\sin x}\,dx$, or is there any other much simpler method to compute this integral?)

Exercise: Compute the following integral:

(1) $\int \frac{1}{2+\sin 4x+\cos 4x}\,dx$.

(2) $\int \frac{1}{3x^2+4}\,dx$.

(3) $\int \frac{1}{2x^2+7x+3}\,dx$.

(4) $\int \frac{1}{2x^2+7x+7}\,dx$.

3 Matrix

3.1 Second method of finding inverse of matrix: Using elementary row operation

To begin with, let's learn three elementary row operations:

i Exchanging two rows:

E.g. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 4 & 5 & 6 & 11 \\ 1 & 2 & 3 & 10 \\ 7 & 8 & 9 & 12 \end{array}\right)$

ii Multiplying a row by a real constant k:

E.g. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right) \xrightarrow{2R_1 \rightarrow R_1} \left(\begin{array}{ccc|c} 2 & 4 & 6 & 20 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right)$

iii Adding a row on another row:

E.g. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right) \xrightarrow{2R_1 + R_2 \rightarrow R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 6 & 9 & 12 & 31 \\ 7 & 8 & 9 & 12 \end{array}\right)$

Use:

i Finding the equation of a system of linear equation. It's what Gaussian elimination does. We don't focus on this part. We just mention a new concept, reduced row echelon form(RREF) (Extracted and modified from Wikipedia page of Row echelon form):

(1) All rows having only zero entries are at the bottom

(2) The leading entry (the left-most non-zero entry) of every non-zero row, is on the right of the leading entry of every row above.

(3) For every non-zero row, the leading entry is 1.

(4) For column containing a leading 1, other entries are 0.

Example of RREF:

(1) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(2) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Example of matrices that are not in RREF:

(1) (Rows having only zero entries are not at the bottom):

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(2) (The leading entry of the third row is on the left of the leading entry of the second row):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(3) (The leading entry of the first and third row is not 1):

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

(4) (The first column containing a leading 1 on the first row, but also contains 10 on the third row):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 10 & 0 & 1 \end{pmatrix}$$

ii Finding inverse of a matrix.

Key: These three operations preservethe singularity of the left matrix, while (i) and (iii) even preserves the determinant of the left matrix.

Example: Use row operation to find the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 4 \end{pmatrix}$

Note that we have

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-R_1+R_2\to R_2,-2R_1+R_3\to R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 1 & 2 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-R_+R_3\to R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3\to R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{-R_3+R_1\to R_1,-3R_3+R_2\to R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -4 & -2 & 3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{-R_2+R_1\to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 1 & -2 \\ 0 & 1 & 0 & -4 & -2 & 3 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

Hence we have $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 1 & -2 \\ -4 & -2 & 3 \\ 1 & 1 & -1 \end{pmatrix}$

Why this method works? Here is a brief motivation:

i Note that every elementary row operation can be regarded as a matrix left-multiplication (You can try to think of it by yourself, or find the Wikipedia page for "elementary row operation"). The R.H.S shows the product of all matrices of elementary row operation done.

ii Using finite number of steps (Why?), we must transform the L.H.S to identity matrix., and R.H.S is the product of all matrices of elementary row operation done, denote it to be L to transform the original matrix A to identity matrix.

iii Then we have $LA = I$. Therefore L is the inverse of A.

Exercise:

i Find the inverse of the following matrices:

(1) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ 2 & 3 & 4 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 21 \end{bmatrix}$

4 Common proof techniques

4.1 Proof by providing counter-examples

Example: Determine whether the following statement is true:

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(x) = 0$, then f attains either local minimum or local maxiumum value at x.

Proof: The statement is false: Let $f(x) = x^3$. Then $f'(0) = 0$ but $f(0)$ is neither local minimum nor local maximum.

4.2 Proof by exhaustion

Example: Show that the remainder of any square number divided by 3 is either 0 or 1. (We said an integer n is divisible by an integer a if there exists an integer k such that $n = ak$.)

Proof. Note that the remainder of every number divided by 3 is either $0, 1$ or 2. (Question: For separating cases, why don't we just divide into two cases-even and odd number?).

Then we separate the possibility of n into three cases:

Case 1: The remainder is 0, Then $n = 3k$ for some integers k.

Then we have

$$\begin{aligned} n^2 &= (3k)^2 \\ &= 9k^2 \\ &= 3(3k^2) \text{ , where } 3k^2 \text{ is an integer.} \end{aligned}$$

Hence n^2 is divisible by 3, therefore remainder of n divided by 3 is 0.

Case 2: The remainder is 1, Then $n = 3k + 1$ for some integers k.

Then we have

$$\begin{aligned} n^2 &= (3k+1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \text{ , where } 3k^2 + 2k \text{ is an integer.} \end{aligned}$$

Therefore remainder of n divided by 3 is 1.

Case 3: The remainder is 2, Then $n = 3k + 2$ for some integers k.

Then we have

$$\begin{aligned} n^2 &= (3k+2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k + 1) + 1 \text{ , where } 3k^2 + 4k + 1 \text{ is an integer.} \end{aligned}$$

Therefore remainder of n divided by 3 is 1.

So combining all cases, we have the remainder of any square number divided by 3 is either 0 or 1.

□

4.3 Proof by contradiction

4.4 If and only if question

Separate the proof into two parts:

Example: Prove that for $\mathbf{v} \in \mathbb{R}^3$, we have $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$. (Dot product is an example of inner product, denote by $< \mathbf{v}, \mathbf{v} >$. We will discuss it in the next tutorial).

In other words, we will say that these two statements are equivalent.

Proof.

($\Longrightarrow$) If $\mathbf{v} \cdot \mathbf{v} = 0$. Then we have

$$(v_1, v_2, v_3) \cdot (v_1, v_2, v_3) = 0$$
$$v_1^2 + v_2^2 + v_3^2 = 0$$

Suppose $v_1 \neq 0$. Then we have $v_1^2 > 0$, thereby $v_1^2 + v_2^2 + v_3^2 \geq v_1^2 + 0 + 0 = v_1^2 > 0$, Contradiction arises. Thereby we must have $v_1 = 0$

Similarly, we have $v_2 = v_3 = 0$.

Hence we have $\mathbf{v} = \mathbf{0}$.

($\Longleftarrow$) Trivially, $\mathbf{0} \cdot \mathbf{0} = (0,0,0) \cdot (0,0,0) = 0 \cdot 0 + 0 \cdot 0 + 0 \cdot 0 = 0$. □

Example of non-equivalent statements:

Statement 1: $x^2 = 1$.

Statement 2: $x = 1$.

Statement 2 implies stamtement 1, but statemene 1 does not implies statement 2 as $x = 1$ or $x = -1$ both leads to the result $x = 1$, so $x = 1$ is not a unique solution of $x^2 = 1$.

5 Span and linear independence

Key: Why do we need to assume vector space is closed under addition and scalar multiplication?

Steps for showing a set A is a basis of the given vector space V:

Step 1: Show that A is a linearly independent set.

Step 2: Show that for all vectors $\overrightarrow{u} \in V$, we can express $\overrightarrow{u}$ as a linear combination of vectors in A

Reminder: Note that basis of a vector space is not unique: Example: Show that $\{(1,1),(0,1)\}$ is a basis of $\mathbb{R}^2$.(Note that $\{(0,1),(0,1)\}$ is also a basis of $\mathbb{R}^2$)

So when answering questions, don't say something like "Let S be the basis of the given vector space V". Writing this means that you are asking us to deduct your marks!

Question: Can we find a basis of spaces containing all non-singular matrix?

Exercise: Determine whether the following form a basis of $\mathbb{R}^3$:

i $\{(1,4,7),(2,5,8),(3,6,9)\}$.

ii $\{(1,4,2),(2,5,3),(3,7,4)\}$.

In Mathematics, we usually do similar things: To understand the properties of a set/system from smaller and basic sets. For instance, in group theory, we can understand the properties of a group from its generating set("spanning set" of a group). In measure theory, we understand the properties of measure and measurable set from intervals.

6 Transformation matrix and change of basis

Example:

i Rotation matrix:

In $\mathbb{R}^2$, if we rotate a $\mathbb{R}^2$ vector anti-clockwisely by θ radian, then we have the associated matrix representation to be

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

ii Reflection matrix:

If we reflect $\mathbb{R}^2$ vector along $x = y$, the associated matrix representation to be

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note that these transformations are **linear**.

Definition 6.1 (Linear map)**.** *A map $T : \mathbb{R}^m \to \mathbb{R}^n$ is said to be **linear** if*

i For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

ii For all $\mathbf{u} \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, we have $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$.

We have some useful lemma from this definition:

Lemma 6.1 (Properties of linear map)**.**

i $T(\mathbf{0}) = \mathbf{0}$.

ii For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, we have $T(\alpha\mathbf{u} + \mathbf{v}) = \alpha T(\mathbf{u}) + T(\mathbf{v})$.

(Easy but will not be examined).

i

$$\begin{aligned} T(\mathbf{0} + \mathbf{0}) &= T(\mathbf{0}) \\ T(\mathbf{0}) + T(\mathbf{0}) &= T(\mathbf{0}) \text{ (By (ii) of definition of linear map)} \\ T(\mathbf{0}) &= \mathbf{0} \text{ (By Cancellation Law of real numbers)} \end{aligned}$$

ii

$$\begin{aligned} T(\alpha\mathbf{u} + \mathbf{v}) &= T(\alpha\mathbf{u}) + T(\mathbf{v}) \text{ (By (i) of definition of linear map)} \\ &= \alpha T(\mathbf{u}) + T(\mathbf{v}) \text{ (By (ii) of definition of linear map)} \end{aligned}$$

□

You may wonder how these matrices "pop out". Let's find a general way of finding the matrix representation of a linear transformation

General way of finding a matrix representation of a linear transformation. In $\mathbb{R}^2$, note that $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$. Note that for every $\mathbf{v} \in \mathbb{R}^2$, we can find $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\mathbf{v} = \alpha_1(1,0) + \alpha_2(0,1)$.

Then for every linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, with $\lfloor \mathbf{a} \rfloor_x = x-$coordinate of $\mathbf{a}$ and $\lfloor \mathbf{a} \rfloor_y = y-$coordinate of $\mathbf{a}$, we have

$$\begin{aligned} T(\mathbf{v}) &= T(\alpha_1(1,0) + \alpha_2(0,1)) \\ &= \alpha_1 T((1,0)) + \alpha_2 T((0,1)) \\ &= \begin{pmatrix} \lfloor T((1,0)) \rfloor_x & \lfloor T((0,1)) \rfloor_x \\ \lfloor T((1,0)) \rfloor_y & \lfloor T((0,1)) \rfloor_y \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \end{aligned}$$

Then we can say that $\begin{pmatrix} \lfloor T((1,0)) \rfloor_x & \lfloor T((0,1)) \rfloor_x \\ \lfloor T((1,0)) \rfloor_y & \lfloor T((0,1)) \rfloor_y \end{pmatrix}$ is the matrix representation of a linear transformation T.

Similarly, for every linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, and $\mathbf{v} = \alpha_1(1,0,0)+\alpha_2(0,1,0)+\alpha_3(0,0,1)$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, we have

$$T(\mathbf{v}) = \begin{pmatrix} \lfloor T((1,0,0)) \rfloor_x & \lfloor T((0,1,0)) \rfloor_x & \lfloor T((0,0,1)) \rfloor_x \\ \lfloor T((1,0,0)) \rfloor_y & \lfloor T((0,1,0)) \rfloor_y & \lfloor T((0,0,1)) \rfloor_y \\ \lfloor T((1,0,0)) \rfloor_z & \lfloor T((0,1,0)) \rfloor_z & \lfloor T((0,0,1)) \rfloor_z \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Then we can say that $\begin{pmatrix} \lfloor T((1,0,0)) \rfloor_x & \lfloor T((0,1,0)) \rfloor_x & \lfloor T((0,0,1)) \rfloor_x \\ \lfloor T((1,0,0)) \rfloor_y & \lfloor T((0,1,0)) \rfloor_y & \lfloor T((0,0,1)) \rfloor_y \\ \lfloor T((1,0,0)) \rfloor_z & \lfloor T((0,1,0)) \rfloor_z & \lfloor T((0,0,1)) \rfloor_z \end{pmatrix}$ is the matrix representation of a linear transformation T.

Similarly, you can find the matrix representation of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$. You can try it by yourself.

So let's go back to find the matrix representation of rotation matrix and reflection along $x = y$:

i Rotation matrix:

From $\cos^2\theta + \sin^2\theta = 1$, we can regard the set $\{(\cos\theta, \sin\theta) : 0 \le \theta < 2\pi\}$ as a circle.

Also, note that $(1,0) = (\cos 0, \sin 0)$ and $(0,1) = (\cos\frac{\pi}{2}, \sin\frac{\pi}{2})$.

Hence we have $T((1,0)) = ((\cos\theta, \sin\theta))$ and $T((0,1)) = \left(\cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right)\right) = (-\sin\theta, \cos\theta)$.

Hence the associated matrix representation is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

ii Reflection along $x = y$:

Note that $T((1,0)) = (0,1)$ and $T((0,1)) = (1,0)$.

Hence the associated matrix representation is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Exercise: Find the associated matrix representation of the following linear transformation:

i In $\mathbb{R}^2$, reflect the vector along the straight line $x = 2y$.

ii In $\mathbb{R}^3$, rotate the vector along $y-$axis (We denote $y-$axis is into/out of the paper, $x-$axis is left/right, $z-$axis is up/down) clockwisely by θ radian, where $\theta \in \mathbb{R}$. (Use right-hand grid rule to find the direction)

Solution:

i Find $T((1, 0))$ and $T((0, 1))$:
Method: Find the line perpendicular to $x = 2y$ and passing through $(1, 0)$: $y = -2x + 2$.
Point of intersection between $y = -2x + 2$ and $x = 2y$: $(0.8, 0.4)$.
Denote the reflected point to be D. Note that the midpoint between D and $(1, 0)$ is $(0.8, 0.4)$.
Therefore, we have $D = T((1, 0)) = (0.6, 0.8)$.

Similarly, the line perpendicular to $x = 2y$ and passing through $(0, 1)$: $y = -2x + 1$.
Point of intersection between $y = -2x + 1$ and $x = 2y$: $(0.4, 0.2)$.
Denote the reflected point to be E. Note that the midpoint between E and $(1, 0)$ is $(0.4, 0.2)$.
Therefore, we have $D = T((1, 0)) = (0.8, -0.6)$.

So the required matrix is $\begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix}$

ii $$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

6.1 Surjectivity and injectivity

Definition 6.2 (Surjectivity and injectivity)**.** *A function $f : A \to B$ is said to be*

i surjective if for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

ii injective if for all $x, y \in A$ such that $f(x) = f(y)$, we have $x = y$.

iii bijective if f is both surjective and injective.

Note that if f is **bijective**, then **inverse of a function** is well-definied and unique, denote it by f^{-1}. (E.g. Inverse of $y = x^3$ is $y = x^{\frac{1}{3}}$.)

Examples of injective map:

i $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$.

ii $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2^x$.

Examples of non-injective map:

i $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$: $f(1) = f(-1) = 1$.

ii $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x,\ f(0) = f(\pi) = 0$.

Examples of surjective map:

i $f : \mathbb{R} \to \mathbb{R},\ f(x) = x$.

ii $f : \mathbb{R} \to [-1, 1],\ f(x) = \sin x$: As $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$

Examples of non-surjective map:

i $f : \mathbb{R} \to \mathbb{R},\ f(x) = 2^x$: As $2^x > 0$ for all $x \in \mathbb{R}$

ii $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$: As $x^2 \geq 0$ for all $x \in \mathbb{R}$

iii $f : \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$: As $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$

In finite dimensional linear map, surjectivity is equivalent to injectivity. (We won't prove it here)

Differentiation map is linear. It is surjective but not injective(why?)

i Surjectivity:
For all continuous $f : [a, b] \to \mathbb{R}$, for $x \in [a, b]$ define $G(x) = f(a) + \int_a^x f(y)\ dy$ (Sometimes we just know the existence of the function G, but we don't know the form of G, unlike secondary school, we may not be able to find the anti-derivative/indefinite integral of a function). Then $G'(x) = f(x)$.

ii Injectivity:
Let $f(x) = x + 1$ and $g(x) = x$. Obviously, we have $f(x) \neq g(x)$ but $f'(x) = 1 = g(x)$ for all $x \in \mathbb{R}$. This means that differential function is not injective.